The range of indecomposability of ultrafilters at successors of singular cardinals

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Set Theory in the United Kingdom 15, 2025

Definition

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- Let us denote the indecomposability range of $\mathcal{U}$ by:

$$\Theta(\mathcal{U}) := \{\eta < \lambda \mid \mathcal{U} \text{ is } \eta\text{-indecomposable}\}$$

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Remark

Notice that Indecomposability of an ultrafilter is a weakening of the notion of completeness in which a limit is guaranteed to exist only for linear intersections.

Motivation

Theorem (Ben-David and Magidor, 1986)

If κ is κ^+ supercompact then there is a generic extansion in which there is an $\aleph_n$-indecomposible ultrafilter on $\aleph_{\omega+1}$ for any $1 < n < \omega$.

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Remark

In the same paper, as a consequence, they showed that $\Box^*_{\aleph_\omega}$ can be obtained. Thus, by that they showed that $\Box^*$ is weaker than $\Box$.

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Remark

The model construction appears in [Magidor, 1977].

A model for global compactness

Theorem (Jirranttikansakul, O. and Rinot)

Let κ *be a supercompact cardinal in* V*, then there is a forcing extension* W*, in which* κ *is inaccessible and for every singular cardinal* $\lambda < \kappa$*, there exists an ultrafilter on* λ^+ *which is* θ*-indecomposable for any regular* $\theta \in (cf(\lambda), \lambda)$*.*

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Remark

Then W_κ (i.e. V_κ^W) is a model of *ZFC* in which for every singular cardinal λ, there exists is an ultrafilter on λ^+ which is θ-indecomposable for any regular $\theta \in (cf(\lambda), \lambda)$.

A model for global compactness

Corollary

Assuming there is a model of ZFC with a supercompact cardinal, then there is a model of ZFC, in which on every successor of a singular cardinal, λ^+, there is a uniform ultrafilter $\mathcal{U}_\lambda$ such that $\Theta(\mathcal{U}_\lambda) = (\mathrm{cf}(\lambda), \lambda)$.

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Theorem (Usuba, 2025)

Let κ be a singular cardinal, and $\mathcal{U}$ a uniform ultrafilter over κ^+. If $\mathcal{U}$ is κ-indecomposable, then there is a cardinal $\eta < \kappa$ such that for every $\mu \in (\lambda, \kappa)$ regular $\mu \notin \Theta(\mathcal{U})$.

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Remark

By Usuba's proposition, it is impossible to get $\mathcal{U}$ on λ^+ such that $(\mathrm{cf}(\lambda), \lambda] \subseteq \Theta(\mathcal{U})$, hence we can't expand the range of indecomposability of $\mathcal{U}_\lambda$ on λ^+ to include λ.

Lower bound for $\Theta(\mathcal{U})$?- Work in progress

In order to answer this question, I want to remind you the Prikry-tree forcing. Let $\mathcal{U}$ be some uniform ultrafilter on κ, regular.

Definition

- Let $\mathbb{P}_{Tree,\mathcal{U}}$ be the collection of all $\mathcal{U}$-trees.

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  A condition $T \in \mathbb{P}_{Tree,\mathcal{U}}$ is a tree $T \subseteq [\kappa]^{<\omega}$ such that
  there is some $t \in T$, denoted by $t = Stem(T)$,
  and such that for all $s \in T$ either $s \leq_T t$ or $t \leq_T s$ and for all $s \geq_T t$
  $Succ_T(s) = \{i < \kappa \mid s + < i > \in T\} \in \mathcal{U}$.

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  $Succ_T(s) = \{i < \kappa \mid s+ < i > \in T\} \in \mathcal{U}$.
- We say that $S \leq T$ iff $S \subseteq T$.
- We say that $S \leq^* T$ iff $S \leq T$ and $Stem(S) = Stem(T)$.

General observations on $\mathbb{P}_{Tree,\mathcal{U}}$

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Lemma

Let $T, S \in \mathbb{P}_{Tree,\mathcal{U}}$ are such that $Stem(S) = Stem(T)$, then $S \cap T$ is a condition and $S \cap T \leq^* S, T$.
Moreover $\mathbb{P}_{Tree,\mathcal{U}}$ has the κ^+- chain condition.

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Remark

The proof only uses the fact that $\mathcal{U}$ is a filter.

$\mathbb{P}_{Tree,\mathcal{U}}$, using the indecomposability range of $\mathcal{U}$

Lemma (*)

Let $T \in \mathbb{P}_{Tree,\mathcal{U}}$ *and* $\dot{\tau}$ *such that* $T \Vdash \dot{\tau} < \check{\alpha}$ *for* α *with* $\mathrm{cf}(\alpha) \in \Theta(\mathcal{U})$.
Then there is $T' \leq^* T$ *and* $\beta < \alpha$ *such that* $T' \Vdash \dot{\tau} < \check{\beta}$.

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Theorem (*)

If $(\rho, \lambda) \subseteq \Theta(\mathcal{U})$, then $\mathbb{P}_{Tree,\mathcal{U}}$ preserves all regular cardinals in (ρ^+, λ).

Preserving regular cardinals in (ρ^+, λ)

Sketch of the proof:

Let $\eta \in (\rho^+, \lambda)$ regular, in order to show that η is preserved we will prove that for all regular $\alpha \in (\rho, \eta)$ all stationary $S \subseteq S^\eta_\alpha$ are preserved.

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$T \Vdash \dot{C} : \mu \to \eta$ is a club s.t. $\forall i \in \dot{C}\, \mathrm{cf}(i) < \check{\mu}$.

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$T \Vdash \dot{C} : \mu \to \eta$ is a club s.t. $\forall i \in \dot{C}\ \mathrm{cf}(i) < \check{\mu}$.
Thus there is some $\alpha \in [\mu, \eta) \cap \Theta(\mathcal{U})$ and then all stationary $S \subseteq S^\eta_\alpha$ are preserved, but then $T \Vdash \dot{C} \cap S = \emptyset$, which is a contradiction.

Preserving regular cardinals in (ρ^+, λ), Sketch

Let $S \subseteq S^\eta_\alpha$ and T such that

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Then by the Lemma, there is some $T' \leq^* T$ and $\gamma < \delta$ such that

$$T' \Vdash \dot{\tau} < \check{\gamma}.$$

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Which is a contradiction. $\square$

Limitations on $\Theta(\mathcal{U})$

Remark

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Proof.

If $G \subseteq \mathbb{P}_{Tree,\mathcal{U}}$ is V generic, then

$$V[G] \models \mathrm{cf}((\lambda^+)^V) = \omega.$$

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This property holds for $\mathcal{U}_\lambda$, the indecomposables on λ^+ in the joint work with Sittinon Jirrattikansakul and Assaf Rinot, as well as the Ben David Magidor ultrafilter on $\aleph_{\omega+1}$.

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This property holds for $\mathcal{U}_\lambda$, the indecomposables on λ^+ in the joint work with Sittinon Jirrattikansakul and Assaf Rinot, as well as the Ben David Magidor ultrafilter on $\aleph_{\omega+1}$. The conjecture, that this is the case for all ultrafilters with such indecomposable range is currently under progress.

Thank You!