A model with an indecomposable ultrafilter at every successor of a singular cardinal

Inbar Oren

The Hebrew University of Jerusalem

120 Years of Choice,
Leeds 2024

This is based on a joint paper with Sittinon Jirattikansakul and Assaf Rinot. [1]

[1] https://papers.assafrinot.com/paper67.pdf

The Main theorem

Theorem

Let κ be a supercompact cardinal in V, then there is a forcing extension W, in which κ is inaccessible and for every singular cardinal $\lambda < \kappa$, there exists an ultrafilter on λ^+ which is θ-indecomposable for any regular $\theta \in (cf(\lambda), \lambda)$.

The Main theorem

Theorem

Let κ be a supercompact cardinal in V, then there is a forcing extension W, in which κ is inaccessible and for every singular cardinal $\lambda < \kappa$, there exists an ultrafilter on λ^+ which is θ-indecomposable for any regular $\theta \in (cf(\lambda), \lambda)$.

Remark

Then W_κ (i.e. V_κ^W) is a model of ZFC in which for every singular cardinal λ, there exists is an ultrafilter on λ^+ which is θ-indecomposable for any regular $\theta \in (cf(\lambda), \lambda)$.

Definition

Suppose $\mathcal{U}$ is an ultrafilter over an infinite cardinal λ.

Definition

Suppose $\mathcal{U}$ is an ultrafilter over an infinite cardinal λ.

- $\mathcal{U}$ is *uniform* iff $|X| = \lambda$ for all $X \in \mathcal{U}$;

Definition

Suppose $\mathcal{U}$ is an ultrafilter over an infinite cardinal λ.

- $\mathcal{U}$ is *uniform* iff $|X| = \lambda$ for all $X \in \mathcal{U}$;
- For a cardinal $\theta < \lambda$, $\mathcal{U}$ is θ*-indecomposable* iff for every partition $\langle X_\alpha \mid \alpha < \theta \rangle$ of λ, there is an $A \in [\theta]^{<\theta}$ such that $\bigcup_{\alpha \in A} X_\alpha \in \mathcal{U}$.

Remark

Notice that the Indecomposability of an ultrafilter is a weakening of the notion of completeness in which a limit is guaranteed to exist only for linear intersections.

Motivation

Theorem (Ben-David and Magidor, 1986)

If κ is κ^+ supercompact then there is a generic extansion in which there is an $\aleph_n$-indecomposible ultrafilter on $\aleph_{\omega+1}$ for any $1 < n < \omega$.

Motivation

Theorem (Ben-David and Magidor, 1986)

If κ is κ^+ supercompact then there is a generic extansion in which there is an $\aleph_n$-indecomposible ultrafilter on $\aleph_{\omega+1}$ for any $1 < n < \omega$.

Remark

In the same paper, as a consequence, they showed that $\square^*_{\aleph_\omega}$ can be obtained. Thus, by that, they showed that $\square^*$ is weaker than $\square$.

Motivation

Theorem (Ben-David and Magidor, 1986)

If κ is κ^+ supercompact then there is a generic extansion in which there is an $\aleph_n$-indecomposible ultrafilter on $\aleph_{\omega+1}$ for any $1 < n < \omega$.

Remark

The model construction appears in [Magidor, 1977].

Generalization

Assume κ is a supercompact cardinal, we will use a forcing notion $\mathbb{R}$ with some nice properties such as:

Generalization

Assume κ is a supercompact cardinal, we will use a forcing notion $\mathbb{R}$ with some nice properties such as:

- κ^{++}-c.c.

Generalization

Assume κ is a supercompact cardinal, we will use a forcing notion $\mathbb{R}$ with some nice properties such as:

- κ^{++}-c.c.
- Factorization: for cofinaly many $\eta < \kappa$ the forcing $\mathbb{R}$ can be factored to $\mathbb{R}_0 \times \mathbb{R}_1$ where:

Generalization

Assume κ is a supercompact cardinal, we will use a forcing notion $\mathbb{R}$ with some nice properties such as:

- κ^{++}-c.c.
- Factorization: for cofinaly many $\eta < \kappa$ the forcing $\mathbb{R}$ can be factored to $\mathbb{R}_0 \times \mathbb{R}_1$ where:
  - $\mathbb{R}_0$ has the η^{++}-c.c.
  - $(\mathbb{R}_1, \leq^*)$ is η^+ closed.

Generalization

Assume κ is a supercompact cardinal, we will use a forcing notion $\mathbb{R}$ with some nice properties such as:

- κ^{++}-c.c.
- Factorization: for cofinaly many $\eta < \kappa$ the forcing $\mathbb{R}$ can be factored to $\mathbb{R}_0 \times \mathbb{R}_1$ where:
  - $\mathbb{R}_0$ has the η^{++}-c.c.
  - $(\mathbb{R}_1, \leq^*)$ is η^+ closed.
- There are projections of $\mathbb{R}$.

Generalization

Assume κ is a supercompact cardinal, we will use a forcing notion $\mathbb{R}$ with some nice properties such as:

- κ^{++}-c.c.
- Factorization: for cofinaly many $\eta < \kappa$ the forcing $\mathbb{R}$ can be factored to $\mathbb{R}_0 \times \mathbb{R}_1$ where:
  - $\mathbb{R}_0$ has the η^{++}-c.c.
  - $(\mathbb{R}_1, \leq^*)$ is η^+ closed.
- There are projections of $\mathbb{R}$.

Remark

In our paper, we analyze several intermediate models between V and the $\mathbb{R}$-generic extension.

Generalization

- In order to obtain this kind of forcing we wanted to use the ideas of (Magidor, 1977) with a Radin-based forcing.

Generalization

- In order to obtain this kind of forcing we wanted to use the ideas of (Magidor, 1977) with a Radin-based forcing.
- A crucial point in obtaining the indecomposability in the final model is to make sure that the quotient forcing is weakly homogeneous.

Generalization

- In order to obtain this kind of forcing we wanted to use the ideas of (Magidor, 1977) with a Radin-based forcing.
- A crucial point in obtaining the indecomposability in the final model is to make sure that the quotient forcing is weakly homogeneous.
- In Magidor's forcing, the quotient is weakly homogeneous once adding guiding generics.

Generalization

- In order to obtain this kind of forcing we wanted to use the ideas of (Magidor, 1977) with a Radin-based forcing.
- A crucial point in obtaining the indecomposability in the final model is to make sure that the quotient forcing is weakly homogeneous.
- In Magidor's forcing, the quotient is weakly homogeneous once adding guiding generics.
- When we replace Magidor's forcing with a Radin-based forcing we need a different notion in order to ensure weak homogeneity of the quotient, that we call *guru*.

Properties of our intermediate model, W.

We work in W, a generic extension of V, and have the following

Properties of our intermediate model, W.

We work in W, a generic extension of V, and have the following

1. κ is inaccessible in W.

Properties of our intermediate model, W.

We work in W, a generic extension of V, and have the following

1. κ is inaccessible in W.
2. For any $\lambda < \kappa$ singular there is a forcing notion $\mathbb{Q}_\lambda$ in W such that in the generic extantion $\mathrm{cf}((\lambda^+)^W) = \mathrm{cf}(\lambda)$.

Properties of our intermediate model, W.

We work in W, a generic extension of V, and have the following

1. κ is inaccessible in W.
2. For any $\lambda < \kappa$ singular there is a forcing notion $\mathbb{Q}_\lambda$ in W such that in the generic extantion $\mathrm{cf}((\lambda^+)^W) = \mathrm{cf}(\lambda)$.
3. There is a canonical cofinal sequence on λ^+: $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$.

Properties of our intermediate model, W.

We work in W, a generic extension of V, and have the following

1. κ is inaccessible in W.
2. For any $\lambda < \kappa$ singular there is a forcing notion $\mathbb{Q}_\lambda$ in W such that in the generic extantion $\mathrm{cf}((\lambda^+)^W) = \mathrm{cf}(\lambda)$.
3. There is a canonical cofinal sequence on λ^+: $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$.
4. For any $\theta < \lambda$ regular in W there is a projection of $\mathbb{Q}_\lambda$, $\mathbb{Q}_{\lambda,\theta}$ that preserves θ, and contains a tail of $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$.

Properties of our intermediate model, W.

We work in W, a generic extension of V, and have the following

1. κ is inaccessible in W.
2. For any $\lambda < \kappa$ singular there is a forcing notion $\mathbb{Q}_\lambda$ in W such that in the generic extantion $\mathrm{cf}((\lambda^+)^W) = \mathrm{cf}(\lambda)$.
3. There is a canonical cofinal sequence on λ^+: $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$.
4. For any $\theta < \lambda$ regular in W there is a projection of $\mathbb{Q}_\lambda$, $\mathbb{Q}_{\lambda,\theta}$ that preserves θ, and contains a tail of $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$.
5. Weak homogeneity.

Weak homogeneity

Moreover : If $q_0, q_1 \in \mathbb{Q}_\lambda$ then there is an automorpism $\Gamma : \lambda^+ \to \lambda^+$ and some $\beta < \lambda^+$ with $\Gamma \restriction \lambda \cup (\beta, \lambda^+) = \text{id}$ such that $\Gamma(q_0) \parallel q_1$. The "stems" of conditions in $\mathbb{Q}_\lambda$ are subsets of λ^+, hence we can extend Γ to a permutation of $\mathbb{Q}_\lambda$.

Fix $\lambda < \kappa$ which is a singular cardinal in W. Let G_λ be a generic for $\mathbb{Q}_\lambda$.

Fix $\lambda < \kappa$ which is a singular cardinal in W. Let G_λ be a generic for $\mathbb{Q}_\lambda$.

Definition

In $W[G_\lambda]$, define a filter $\mathcal{F}_\lambda$ over λ^+ by:
$A \in \mathcal{F}_\lambda$ iff A contains a tail of the cofinal squence inturduced by G_λ.

Fix $\lambda < \kappa$ which is a singular cardinal in W. Let G_λ be a generic for $\mathbb{Q}_\lambda$.

Definition

In $W[G_\lambda]$, define a filter $\mathcal{F}_\lambda$ over λ^+ by:
$A \in \mathcal{F}_\lambda$ iff A contains a tail of the cofinal squence inturduced by G_λ.

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

We aim to find $F^* \in W$ such that $F^* = \mathcal{F}_\lambda \cap W$.
Define $F^* \in W$ as the collection of X for which there is a $\mathbb{Q}_\lambda$-name $\dot{X}$ (that is forced to be in $\dot{W}$), and some $p \in \mathbb{Q}_\lambda$ such that

$$p \Vdash_{\mathbb{Q}_\lambda} "\dot{X} \in \dot{\mathcal{F}}_\lambda".$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

We aim to find $F^* \in W$ such that $F^* = \mathcal{F}_\lambda \cap W$.
Define $F^* \in W$ as the collection of X for which there is a $\mathbb{Q}_\lambda$-name $\dot{X}$ (that is forced to be in $\dot{W}$), and some $p \in \mathbb{Q}_\lambda$ such that

$$p \Vdash_{\mathbb{Q}_\lambda} "\dot{X} \in \dot{\mathcal{F}}_\lambda".$$

We first check that F^* is well-defined.
Let $X \in W$ with an $\mathbb{Q}_\lambda$-name $\dot{X}$ which is invariant under automorphisms of λ^+ and has $p_0, p_1 \in \mathbb{Q}_\lambda$ with

$$p_0 \Vdash "\dot{X} \in \dot{\mathcal{F}}_\lambda", \quad \text{and} \quad p_1 \Vdash "\dot{X} \notin \dot{\mathcal{F}}_\lambda".$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

We first check that F^* is well-defined.
Let $X \in W$ with an $\mathbb{Q}_\lambda$-name $\dot{X}$ which is invariant under automorphisms of λ^+ and has $p_0, p_1 \in \mathbb{Q}_\lambda$ with

$$p_0 \Vdash "\dot{X} \in \dot{\mathcal{F}}_\lambda", \quad \text{and} \quad p_1 \Vdash "\dot{X} \notin \dot{\mathcal{F}}_\lambda".$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

We first check that F^* is well-defined.
Let $X \in W$ with an $\mathbb{Q}_\lambda$-name $\dot{X}$ which is invariant under automorphisms of λ^+ and has $p_0, p_1 \in \mathbb{Q}_\lambda$ with

$$p_0 \Vdash "\dot{X} \in \dot{\mathcal{F}}_\lambda", \quad \text{and} \quad p_1 \Vdash "\dot{X} \notin \dot{\mathcal{F}}_\lambda".$$

By the weakly homogeneous Lemma, there is $\Gamma : \lambda^+ \to \lambda^+$ with $\beta < \lambda^+$ such that

$$\Gamma(p_0) \parallel p_1.$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

By the weakly homogeneous Lemma, there is $\Gamma : \lambda^+ \to \lambda^+$ with $\beta < \lambda^+$ such that

$$\Gamma(p_0) \parallel p_1.$$

But since $\Gamma \restriction (\beta, \lambda^+) = \mathrm{id}$, it does not change $\dot{\mathcal{F}}_\lambda$, hence

$$\Gamma(p_0) \Vdash \dot{X} \in \dot{\mathcal{F}}_\lambda.$$

Which is a contradiction to $\Gamma(p_0) \parallel p_1$.

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

But since $\Gamma \restriction (\beta, \lambda^+) = \mathrm{id}$, it does not change $\dot{\mathcal{F}}_\lambda$, hence

$$\Gamma(p_0) \Vdash \dot{X} \in \dot{\mathcal{F}}_\lambda.$$

Which is a contradiction to $\Gamma(p_0) \parallel p_1$.
Since otherwise there is $q \leq p_0, p_1$ such that

$$q \Vdash \dot{X} \in \dot{\mathcal{F}}_\lambda, \quad \text{and} \quad q \Vdash \dot{X} \notin \dot{\mathcal{F}}_\lambda.$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

But since $\Gamma \restriction (\beta, \lambda^+) = \mathrm{id}$, it does not change $\dot{\mathcal{F}}_\lambda$, hence

$$\Gamma(p_0) \Vdash \dot{X} \in \dot{\mathcal{F}}_\lambda.$$

Which is a contradiction to $\Gamma(p_0) \parallel p_1$.
We now show that $F^* = \mathcal{F}_\lambda \cap W$:

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

But since $\Gamma \restriction (\beta, \lambda^+) = \mathrm{id}$, it does not change $\dot{\mathcal{F}}_\lambda$, hence

$$\Gamma(p_0) \Vdash \dot{X} \in \dot{\mathcal{F}}_\lambda.$$

Which is a contradiction to $\Gamma(p_0) \parallel p_1$.
We now show that $F^* = \mathcal{F}_\lambda \cap W$:
Let $X \in F^*$.

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

We now show that $F^* = \mathcal{F}_\lambda \cap W$:

Let $X \in F^*$.

Then by the definition of F^* there is $p \in \mathbb{Q}_\lambda$, such that

$$p \Vdash_{\mathbb{Q}_\lambda} "\dot{X} \in \dot{\mathcal{F}}_\lambda".$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

We now show that $F^* = \mathcal{F}_\lambda \cap W$:

Let $X \in F^*$.

Then by the definition of F^* there is $p \in \mathbb{Q}_\lambda$, such that

$$p \Vdash_{\mathbb{Q}_\lambda} "\dot{X} \in \dot{\mathcal{F}}_\lambda".$$

Let $p' \in G_\lambda$ that decides the statement $\dot{X} \in \dot{\mathcal{F}}_\lambda$. Then, by weak homogeneity, there is $\Gamma : \lambda^+ \to \lambda^+$ with some $\beta < \lambda^+$ such that

$$\Gamma(p) \parallel p'.$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

$$p \Vdash_{\mathbb{Q}_\lambda} "\dot{X} \in \dot{\mathcal{F}}_\lambda".$$

Let $p' \in G_\lambda$ that decides the statement $\dot{X} \in \dot{\mathcal{F}}_\lambda$. Then, by weak homogeneity, there is $\Gamma : \lambda^+ \to \lambda^+$ with some $\beta < \lambda^+$ such that

$$\Gamma(p) \parallel p'.$$

Hence,

$$p' \Vdash "\dot{X} \in \dot{\mathcal{F}}_\lambda".$$

Claim

$\mathcal{F}_\lambda \cap W \in W$.

Proof.

Let $p' \in G_\lambda$ that decides the statement $\dot{X} \in \dot{\mathcal{F}}_\lambda$. Then, by weak homogeneity, there is $\Gamma : \lambda^+ \to \lambda^+$ with some $\beta < \lambda^+$ such that

$$\Gamma(p) \parallel p'.$$

Hence,

$$p' \Vdash "\dot{X} \in \dot{\mathcal{F}}_\lambda".$$

Thus,

$$X \in \mathcal{F}_\lambda \cap W.$$

□

We continue to work in W.

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.
Then $\mathcal{U}_\lambda$ is a uniform ultrafilter over λ^+.

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.
Then $\mathcal{U}_\lambda$ is a uniform ultrafilter over λ^+.

Lemma

Let $\theta \in (\mathrm{cf}(\lambda), \lambda)$ be some ordinal that is a regular cardinal in $W[G_\lambda]$. Then in W $\mathcal{U}_\lambda$ is θ-indecomposable.

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.
Then $\mathcal{U}_\lambda$ is a uniform ultrafilter over λ^+.

Lemma

Let $\theta \in (\mathrm{cf}(\lambda), \lambda)$ be some ordinal that is a regular cardinal in $W[G_\lambda]$. Then in W $\mathcal{U}_\lambda$ is θ-indecomposable.

Proof.

□

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.
Then $\mathcal{U}_\lambda$ is a uniform ultrafilter over λ^+.

Lemma

Let $\theta \in (\mathrm{cf}(\lambda), \lambda)$ be some ordinal that is a regular cardinal in $W[G_\lambda]$. Then in W $\mathcal{U}_\lambda$ is θ-indecomposable.

Proof.

Let $\langle A_i \mid i < \theta \rangle$ be a partition of λ^+.

□

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.
Then $\mathcal{U}_\lambda$ is a uniform ultrafilter over λ^+.

Lemma

Let $\theta \in (\mathrm{cf}(\lambda), \lambda)$ be some ordinal that is a regular cardinal in $W[G_\lambda]$. Then in W $\mathcal{U}_\lambda$ is θ-indecomposable.

Proof.

Let $\langle A_i \mid i < \theta \rangle$ be a partition of λ^+.
In $W[G_\lambda]$, let $\eta(j)$ be the unique η such that $\lambda_j \in A_\eta$, for $j < \mathrm{cf}(\lambda)$.

□

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.
Then $\mathcal{U}_\lambda$ is a uniform ultrafilter over λ^+.

Lemma

Let $\theta \in (\mathrm{cf}(\lambda), \lambda)$ be some ordinal that is a regular cardinal in $W[G_\lambda]$. Then in W $\mathcal{U}_\lambda$ is θ-indecomposable.

Proof.

Let $\langle A_i \mid i < \theta \rangle$ be a partition of λ^+.
In $W[G_\lambda]$, let $\eta(j)$ be the unique η such that $\lambda_j \in A_\eta$, for $j < \mathrm{cf}(\lambda)$.
Since $\mathrm{cf}(\lambda) < \theta$, there is $\alpha < \theta$ such that for all $j < \mathrm{cf}(\lambda)$, $\eta(j) < \alpha$.

□

We continue to work in W.
Let $\mathcal{U}_\lambda$ be an ultrafilter on λ^+ extending $\mathcal{F}_\lambda$.
Then $\mathcal{U}_\lambda$ is a uniform ultrafilter over λ^+.

Lemma

Let $\theta \in (\mathrm{cf}(\lambda), \lambda)$ be some ordinal that is a regular cardinal in $W[G_\lambda]$. Then in W $\mathcal{U}_\lambda$ is θ-indecomposable.

Proof.

Let $\langle A_i \mid i < \theta \rangle$ be a partition of λ^+.
In $W[G_\lambda]$, let $\eta(j)$ be the unique η such that $\lambda_j \in A_\eta$, for $j < \mathrm{cf}(\lambda)$.
Since $\mathrm{cf}(\lambda) < \theta$, there is $\alpha < \theta$ such that for all $j < \mathrm{cf}(\lambda)$, $\eta(j) < \alpha$.
Hence, $\bigcup_{i<\alpha} A_i \in \mathcal{F}_\lambda \cap W$, so $\bigcup_{i<\alpha} A_i \in \mathcal{U}_\lambda$. □

Now, for any $\theta \in (\mathrm{cf}(\lambda), \lambda)$ which is regular in W, we want to show that $\mathcal{U}_\lambda$ is θ-indecomposable.

Now, for any $\theta \in (\mathrm{cf}(\lambda), \lambda)$ which is regular in W, we want to show that $\mathcal{U}_\lambda$ is θ-indecomposable.

Definition

Let $\theta < \lambda$ such that λ is singular in W and θ is regular in W.
Let $G_{\lambda,\theta}$ be the generic filter generated by the projection of $\mathbb{Q}_\lambda$ to $\mathbb{Q}_{\lambda,\theta}$.

Now, for any $\theta \in (\mathrm{cf}(\lambda), \lambda)$ which is regular in W, we want to show that $\mathcal{U}_\lambda$ is θ-indecomposable.

Definition

Let $\theta < \lambda$ such that λ is singular in W and θ is regular in W.
Let $G_{\lambda,\theta}$ be the generic filter generated by the projection of $\mathbb{Q}_\lambda$ to $\mathbb{Q}_{\lambda,\theta}$.
In $W[G_{\lambda,\theta}]$, define a filter $\mathcal{F}_{\lambda,\theta}$ over λ^+ by:
$X \in \mathcal{F}_{\lambda,\theta}$ iff X containes a tail of the generic cofinal sequence.

Notice that $W[G_{\lambda,\theta}] \cap \mathcal{F}_\lambda = \mathcal{F}_{\lambda,\theta}$ hence $W \cap \mathcal{F}_\lambda = W \cap \mathcal{F}_{\lambda,\theta}$.
Therefore $\mathcal{U}_\lambda$ extends $\mathcal{F}_{\lambda,\theta}$.
We can show that $\mathcal{U}_\lambda$ is θ-indecomposable in a similar manner to the previous Lemma (by working in $W[G_{\lambda,\theta}]$).

Notice that $W[G_{\lambda,\theta}] \cap \mathcal{F}_\lambda = \mathcal{F}_{\lambda,\theta}$ hence $W \cap \mathcal{F}_\lambda = W \cap \mathcal{F}_{\lambda,\theta}$.
Therefore $\mathcal{U}_\lambda$ extends $\mathcal{F}_{\lambda,\theta}$.
We can show that $\mathcal{U}_\lambda$ is θ-indecomposable in a similar manner to the previous Lemma (by working in $W[G_{\lambda,\theta}]$).

Lemma

In W, for every $\theta \in (\mathrm{cf}(\lambda), \lambda)$ that is regular in W, $\mathcal{U}_\lambda$ is θ-indecomposable. □

The Main theorem

Theorem

In $V' := V_\kappa^W$, *for every* λ *which is singular, there is an ultrafilter on* λ^+ *which is* θ*-indecomposable for any regular* $\theta \in (\mathrm{cf}(\lambda), \lambda)$.

The Main theorem

Theorem

In $V' := V_\kappa^W$, for every λ which is singular, there is an ultrafilter on λ^+ which is θ-indecomposable for any regular $\theta \in (\mathrm{cf}(\lambda), \lambda)$.

Proof of the main theorem.

Since κ is inaccessible in W, $V_\kappa^W = V'$ is a model of ZFC. Hence by the previous lemmas for every λ which is singular in V', $\mathcal{U}_\lambda$ is an uniform ultrafilter on λ^+ which is θ-indecomposable for any regular $\theta \in (\mathrm{cf}(\lambda), \lambda)$. □

Remark

Similarly, we can get $\mathcal{U}'_\lambda$ an ultrafilter on $\mathcal{P}_\lambda(\lambda^+)$ such that for any regular $\theta \in (\text{cf}(\lambda), \lambda)$, $|ult(\theta, \mathcal{U}'_\lambda)| = \theta$.

Applications

Definition

Let U be a uniform ultrafilter on η. A *base* for η is a collection $\mathcal{B} = \langle A_\gamma \mid \gamma < \tau \rangle$ of sets in U such that for any $A \in U$, there is a γ such that $A_\gamma \subseteq^* A$.

$$\chi(U) := \min\{|\mathcal{B}| \mid \mathcal{B} \text{ is a base for } U\}.$$

$$\mathfrak{u}(\eta) = \min\{\chi(U) \mid U \text{ is a uniform ultrafilter on } \eta\}.$$

Definition

Let $G := (V, E)$ be a graph of η vertices. The chromatic number of G is the minimal θ such that there is a colouring $c : \theta \to V$ with no two adjacent vertices of the same colour.

Applications

The model W appears to be a combinatorial rich model, with applications such as:

1. Small ultrafilter number at a successor of a given singular: Let λ be a singular cardinal in W, hence $\mathcal{U}_\lambda$ is a θ-indecomposible ultrafilter on λ^+ for all regular $\theta \in (\mathrm{cf}(\lambda, \lambda)$. Let $\mu = \aleph_{\lambda+\mathrm{cf}(\lambda)^+}$, and H be $\mathrm{Add}(\aleph_0, \mu)$-generic over W. In $W[H]$, by Theorem 7 of [Raghavan and Shelah,2020], we have:
$$\mathfrak{u}_{\lambda^+} \leq \mu < 2^{\lambda^+}.$$
2. In W, for any singular λ, any graph $G = (\lambda^+, E)$, and regular $\theta \in (\mathrm{cf}(\lambda), \lambda)$, if every subgraph of G of size less than λ has chromatic number at most θ, then G has chromatic number at most θ.

Thank You!